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A mixed manna contains *goods* (that everyone likes) and *bads* (that everyone dislikes), as well as items that are goods to some agents, but bads or satiated to others.

If all items are goods and utility functions are *homogeneous of degree 1* and concave (and monotone), the competitive division maximizes the Nash product of utilities (Gale–Eisenberg): hence it is *welfarist* (determined by the set of feasible utility profiles), unique, continuous, and easy to compute.

We show that the competitive division of a mixed manna is still welfarist. If the zero utility profile is Pareto dominated, the competitive profile is strictly positive and still uniquely maximizes the product of utilities. If the zero profile is unfeasible (for instance, if all items are bads), the competitive profiles are strictly negative and are the critical points of the product of *disutilities* on the efficiency frontier. The latter allows for multiple competitive utility profiles, from which no single-valued selection can be continuous or *resource monotonic*.

Thus the implementation of competitive fairness under linear preferences in interactive platforms like SPLIDDIT will be more difficult when the manna contains bads that overwhelm the goods.

KEYWORDS: Fair division, goods, bads, competitive equilibrium, Nash product, Gale Eisenberg.

1. INTRODUCTION

THE LITERATURE ON FAIR DIVISION OF PRIVATE COMMODITIES, with few exceptions discussed in Section 2, focuses almost exclusively on the distribution of disposable commodities, that is, desirable *goods* like a cake (Steinhaus (1948)), family heirlooms (Pratt and Zeckhauser (1990)), the assets of divorcing partners (Brams and Taylor (1996)), office space between co-workers, seats in overdemanded business school courses (Sönmez and Ünver (2010), Budish and Cantillon (2010)), computing resources in peer-to-peer platforms (Ghodsi, Zaharia, Hindman, Konwinski, Shenker, and Stoica (2011)), and so on.

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Many important fair division problems involve *bads* (nondisposable items generating disutility): family members distribute house chores, workers divide job shifts (Budish (2011)) like teaching loads, cities divide noxious facilities, managers allocate cuts within the firm, and so on. Moreover the bundle we must divide (the *manna*) often contains the two types of items: dissolving a partnership involves distributing its assets as well as its liabilities, some teachers relish certain classes that others loathe, the land to be divided may include polluted as well as desirable areas, and so on. And the manna may contain items, such as shares in risky assets or hours of babysitting, over which preferences are single-peaked without being monotone, so they will not qualify as either good or bad: they are *satisfiable* items. Of course each item may be a good to some agents, a bad to others, and satisfiable to yet other agents. We will speak of a *mixed manna*. Our paper is, to the best of our knowledge, the first to address the interplay of these different types of items in the fair division problem.

To see why it is genuinely more complicated to divide a mixed rather than a good or a bad manna, consider the popular fairness test of *egalitarian equivalence* (EE) due to Pazner and Schmeidler (1978). A division of the manna is EE if everyone is indifferent between her share and some common reference share: with mixed items this property may well be incompatible with efficiency.¹ The news is much better for the division favored by microeconomists for over four decades (Varian (1974)): the *competitive equilibrium with equal incomes* (here, competitive division for short). Existence is guaranteed when preferences are convex, continuous but not necessarily monotonic, and possibly satiated; see, for example, Shafer and Sonnenschein (1975) and Mas-Colell (1982). And this division retains the key normative properties of efficiency, no envy, and core stability from equal initial endowments.

A striking result by Gale, Eisenberg, and others (Gale (1960), Eisenberg (1961), Chipman (1974), Shafer and Sonnenschein (1993)) shows that in the subdomain of utilities *homogeneous of degree 1* (1-homogeneous for short) as well as concave and continuous, the competitive division of goods obtains by simply maximizing the product of individual utilities. This is remarkable for three reasons. First the “resourcist” concept of competitive division guided by price balancing Walrasian demands, has an equivalent “welfarist” interpretation as the Nash bargaining solution of the feasible utility set. Second, the competitive utility profile is unique because by the latter definition it solves a strictly convex optimization program; it is also computationally easy to find and continuous with respect to the parameters of individual utilities (Vazirani (2005), Megiddo and Vazirani (2007)); these properties all fail under general Arrow–Debreu preferences. Finally the result is broadly applicable because empirical work relies mostly on 1-homogeneous utilities, which include additive, Cobb–Douglas, constant elasticity of substitution (CES), Leontief, and their positive linear combinations. So the Gale–Eisenberg theorem is arguably the most compelling practical vindication of the competitive approach to the fair division of goods.

Main Result

We generalize this result to the division of a mixed manna when utilities are concave, continuous, and 1-homogeneous, but not necessarily monotonic. We show that the wel-

¹Two agents 1 and 2 share (one unit of) two items a and b , and their utilities are linear: $u_1(z_1) = z_{1a} - 2z_{1b}$; $u_2(z_2) = -2z_{2a} + z_{2b}$. The only efficient allocation gives a to 1 and b to 2. In an EE allocation (z_1, z_2) there is some $y \geq 0$ such that $u_i(z_i) = u_i(y)$ for $i = 1, 2$. This implies $u_1(z_1) + u_2(z_2) = -(y_a + y_b)$ so that z is not efficient.

farist interpretation of the competitive division is preserved: the set of feasible utility profiles is still all we need to know to identify the competitive utility profiles (those associated with a competitive division of the items), and they are still related to the product of utilities or disutilities. A description of our main result, Theorem 1 in Section 4, follows.

We identify a partition of all division problems in three types, determined by a very simple welfarist property. Keep in mind that, by 1-homogeneity of utilities, the zero of utilities corresponds to the state of the world without any manna to divide. Call an agent *attracted* if there is a share of the manna giving her strictly positive utility, and *repulsed* if there is none, that is to say, zero is her preferred share.

If it is feasible to give a positive utility to all attracted agents, and zero to all repulsed ones, we call this utility profile *positive* and speak of a positive problem. Problems of this type include those with *only goods*, but also all those where, intuitively, the goods can overwhelm the bads. Then the competitive utility profile is positive and maximizes the product of the attracted agents' utilities over positive profiles; as in Gale–Eisenberg, this utility profile is unique and easy to compute. Also, upon arrival of the manna, everyone's utility increases (at least weakly).

By contrast, if the efficiency frontier contains allocations where *everyone* gets a strictly negative utility, we call the problem *negative*. The division of only bads is a negative problem, but this class also includes all problems with not enough goods to overwhelm the bads. Then the competitive utility profiles are the *critical points* (for instance, local maxima or minima) of the product of all *disutilities* on the intersection of the efficiency frontier with the (strictly) negative orthant: in particular, the arrival of the manna implies a (strict) utility loss for everyone.

Finally the *null* problems are those knife-edge cases where the zero utility profile is efficient: it is then the unique competitive utility profile, and the arrival of the manna has no impact on welfare.

Beyond the above similarities of the competitive approach to fair division in positive and negative problems, there are also important differences.

First, computing the efficient disutility profiles critical for the product of disutilities is no longer a convex program, so we expect computational difficulties in problems with many agents and/or items. Second, negative problems routinely have multiple competitive allocations with distinct utility profiles. We show, for instance, that the number of competitive divisions can grow at least exponentially in the (minimum of the) number of agents and goods. Third, any single-valued selection from the competitive correspondence has unpalatable features described below.

Applications to Practical Fair Division

Our results have implications for the user-friendly internet platforms like SPLIDDIT (www.spliddit.org/) or ADJUSTED WINNER (www.nyu.edu/projects/adjustedwinner/), computing fair outcomes in a variety of problems including the division of manna. Visitors to these sites must distribute 1000 points over the different items, and these “bids” are interpreted as fixed marginal utilities for goods or marginal disutilities for bads. Thus a participants *must* report linear preferences (additive utilities), a fairly manageable task if complementarities between items can be ignored. Thousands of visitors have used SPLIDDIT since November 2014, fully aware of the interpretation of their bids (Goldman and Procaccia (2014)). This is proof enough that the linearity assumption is often acceptable in practical division problems.

SPLIDDIT proposes the competitive solution to divide goods, but for the division of tasks it reverts to the egalitarian equivalent solution mentioned in the second paragraph.

That solution may generate envy in problems with three or more agents, and has several other undesirable features explained in Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaya (2016). Moreover, as mentioned above, the EE approach collapses for a mixed manna. So it is tempting to recommend instead the competitive approach for the division of a mixed manna.

Additive utilities are the simplest domain to which our result applies; therefore, we expect multiple competitive allocations when dividing bads and, more generally, in negative problems. Multiplicity is a useful qualitative insight when the interpretation of the equilibrium is descriptive, but our prescriptive approach looks for unambiguous answers to the fair division problem.

If a negative problem involves only two agents with additive utilities, there are several natural ways to single out a particular competitive utility profile. The set of Pareto optimal (dis)utility profiles is a line made of several segments, along which sit (generically) an odd number of competitive profiles, so we can, for instance, pick the “median” profile. In a (necessarily negative) problem with two bads, the set of Pareto optimal and envy-free allocations is similarly a line connecting several segments, and it contains (generically) an odd number of competitive allocations, so the median allocation is again a reasonable compromise. For general problems, we propose a simple but not particularly compelling selection that *maximizes* the product of *disutilities* on the strictly negative part of the efficiency frontier.

However it turns out that in negative problems, *any* single-valued selection from the set of efficient and non-*envious* divisions (a much larger set than the competitive correspondence) has two undesirable features that do not affect positive problems. First, it is discontinuous in the parameters of the problem (the profile of additive utilities): Proposition 1 in Section 7. Second, it violates the familiar *resource monotonicity* (RM) axiom, requiring solidarity in individual welfares when the manna improves: if we increase the amount of a good or decrease that of a bad, everyone should benefit at least weakly.² Moreover the latter impossibility result holds if we replace no envy by the much weaker *fair share guarantee* property:³ Proposition 2 in Section 7.

We conclude that the practical implementation of competitive division, on these internet platforms or elsewhere, is feasible for negative problems but not as palatable as for positive problems.

Turning Bads Into Goods?

It is somewhat counterintuitive (it certainly surprised us) that dividing bads competitively proves so different from dividing goods. Indeed the division of bads can be turned into that of goods in the same way as we view labor as the negative of leisure. Say that we must allocate 5 hours of a painful job *a* among three agents. Working 2 hours on *a* is the same as being exempt from *a* for 3 hours, so the distribution of 5 hours of *a* is equivalent to that of 10 hours of “*a*-exemption” between the three agents. Repeating this for each job, the bad manna (the jobs) becomes a good manna (the exemptions). But in the new problem, no agent can eat more than 5 hours of *a*-exemption, and these additional capacity constraints prevent us from applying the Gale–Eisenberg theorem. It is well known

²RM has been applied to many different resource allocation problems with production and/or indivisibilities. See the recent survey by Thomson (2010).

³That is, no one is worse off than by consuming a $\frac{1}{n}$ th share of every item. Equal split plays the role of the disagreement option in bargaining.

that capacity constraints on individual consumption enlarge the set of competitive allocations.

Contents

After reviewing the literature in Section 2 and defining the model in Section 3, we state our main result in Section 4. We illustrate the multiplicity issue in a series of problems with additive utilities in Section 5; most of the examples involve two agents or two items. For negative problems with additive utilities, Section 6 defines the single-valued competitive rules mentioned above; then Section 7 state two impossibility results. All substantial proofs are given in Section 8.

2. RELATED LITERATURE

Steinhaus' (1948) "cake division" describes the "good" cake as a compact euclidean set, and assumes linear preferences represented by atomless measures over the cake. This model contains ours (for goods) as the special case where the measures have piecewise constant densities. Segal-Halevi and Sziklai (2015) show that the competitive allocations are still the Nash product maximizer, hence generalizing Eisenberg–Gale. They also show that the competitive rule is resource monotonic. The cake-division literature pays some attention to the division of a bad cake to prove the existence of envy-free divisions of the cake (Su (1999), Peterson and Su (2002), and Heydrich and van Stee (2015)) or to examine how the classic algorithms by cuts and queries can or cannot be adapted to this case (Brams and Taylor (1996) and Robertson and Webb (1998)). It does not discuss the competitive rule for a bad cake or a mixed cake. Following our work, Segal-Halevi (2017) discusses the existence of non-*envious* division of a cake with good and bad parts.

Our paper Bogomolnaia et al. (2016) (hereafter the companion paper) focuses on the contrast between "all goods manna" and "all bads manna" problems in the additive domain. It compares systematically the competitive and egalitarian equivalent division rules, vindicating the former rule by additional independence and monotonicity properties. It contains the proof of some of the results stated here in Sections 5 and 7.

The recent work in computational social choice discusses extensively the fair division of goods (see the survey Brandt, Conitzer, Endriss, Lang, and Procaccia (2016)), recognizing the practical convenience of additive utilities and the conceptual advantages of the competitive solution in that domain (see Moulin (2003) and Vazirani (2005)). For instance, Megiddo and Vazirani (2007) show that the competitive utility profile depends continuously upon the rates of substitution and the total endowment; Jain and Vazirani (2010) show that it can be computed in time polynomial in the dimension $n + m$ of the problem (number of agents and of goods).

The fair division of *indivisible goods* with additive utilities is a much studied variant of the standard model. The maximization of the Nash product loses its competitive interpretation and becomes hard to compute (Lee (2017)); however, it is envy-free "up to at most one object" (Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang (2016)) and can be efficiently approximated for many utility domains (Cole and Gkatzelis (2015), Anari, Gharan, Saberi, and Singh (2016), Anari, Mai, Gharan, and Vazirani (2016), and Cole, Devanur, Gkatzelis, Jain, Mai, Vazirani, and Yazdanbod (2016)). Also Budish (2011) approximates the competitive allocation in problems with a large number of copies of several good types by allowing some flexibility in the number of available copies.

The probabilistic assignment of goods with von Neumann–Morgenstern utilities is another fair division problem with linear and satiated preferences where Hylland and Zeckhauser (1979) and the subsequent literature recommend (a version of) the competitive rule (e.g., He, Miralles, Pycia, and Yan (2015)). That rule is no longer related to the maximization of the product of utilities.

The purely welfarist axiomatic discussion of nonconvex bargaining problems identifies the set of critical points of the Nash product among efficient utility profiles as a natural generalization of the Nash solution: Herrero (1989) and Serrano and Shimomura (1998). This solution stands out also in the rationing model of Mariotti and Villar (2005) where we divide utility losses instead of gains. The latter is closer in spirit to our results for the division of bads.

3. THE MODEL

Notation: We write \mathbb{R}_+^X , \mathbb{R}_-^X , \mathbb{R}_{++}^X , and \mathbb{R}_-^X for, respectively, the nonnegative, nonpositive, strictly positive, and strictly negative vectors in \mathbb{R}^X .

The set of agents is N , that of items is A ; both are finite. The domain $\mathcal{H}(A)$ consists of all preferences on \mathbb{R}_+^A represented by a real-valued utility function v on \mathbb{R}_+^A that is concave, continuous, and 1-homogeneous: $v(\lambda y) = \lambda v(y)$ for all $\lambda \geq 0$, $y \in \mathbb{R}_+^A$. It is easily checked that if two such utility functions represent the same preference, they differ by a positive multiplicative constant. All our definitions and results are purely ordinal, that is, independent of the choice of the utility representations; we abuse language by speaking of “the utility function v in $\mathcal{H}(A)$.”

The graph of a concave and continuous function v on \mathbb{R}_+^A is the envelope of its supporting hyperplanes; therefore, it takes the form $v(y) = \min_{k \in K} \{\alpha_k \cdot y + \beta_k\}$ for some $\alpha_k \in \mathbb{R}^A$, $\beta_k \in \mathbb{R}$, and a possibly infinite set K . It is easy to see that v is also 1-homogeneous if and only if we can choose $\beta_k = 0$ for all k . So the simplest utilities in $\mathcal{H}(A)$ are additive, $v(y) = \alpha \cdot y$, and piecewise linear. For instance, $A = \{a, b\}$ and $v(y) = \min\{y_a + y_b, 4y_a - y_b, 4y_b - y_a\}$, of which the indifference contours are represented on Figure 1, left. Note that this utility is not globally satiated, but for fixed y_b it is satiated at $y_a = y_b$. A smooth nonlinear and nonmonotonic function in $\mathcal{H}(A)$ is, for example, $v(y) = y_b \ln\{\frac{y_a}{y_b} + \frac{1}{2}\}$, represented in Figure 1, right.

A fair division problem is $\mathcal{P} = (N, A, u, \omega)$, where $u \in \mathcal{H}(A)^N$ is the profile of utility functions and $\omega \in \mathbb{R}_+^A$ is the manna; we assume $\omega_a > 0$ for all a .

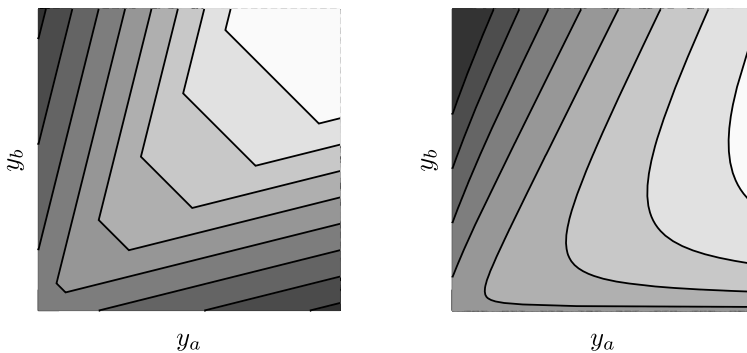


FIGURE 1.—Examples of indifference contours.

A *feasible allocation* (or simply an *allocation*) is $z \in \mathbb{R}_+^{N \times A}$ such that $\sum_N z_{ia} = \omega_a$ for all a or, in a more compact notation, $z_N = \omega$. The corresponding utility profile is $U \in \mathbb{R}^N$, where $U_i = u_i(z_i)$. Let $\mathcal{F}(N, A, \omega)$ be the set of feasible allocations, and let $\mathcal{U}(\mathcal{P})$ be the corresponding set of utility profiles. We always omit \mathcal{P} or N, A if this creates no confusion.

We call a feasible utility profile U *efficient* if it is not Pareto dominated;⁴ a feasible allocation is efficient if it implements an efficient utility profile.

DEFINITION 1: Given problem \mathcal{P} , a competitive division is a triple $(z \in \mathcal{F}, p \in \mathbb{R}^A, \beta \in \{-1, 0, +1\})$, where z is the competitive allocation, p is the competitive price, and β is the individual budget. The allocation z is feasible and each z_i maximizes i 's utility in the budget set $B(p, \beta) = \{y_i \in \mathbb{R}_+^A \mid p \cdot y_i \leq \beta\}$:

$$z_i \in d_i(p, \beta) = \arg \max_{y_i \in B(p, \beta)} \{u_i(y_i)\}. \quad (1)$$

Moreover, z_i minimizes i 's wealth in her demand set

$$z_i \in \arg \min_{y_i \in d_i(p, \beta)} \{p \cdot y_i\}. \quad (2)$$

We write $CE(\mathcal{P})$ for the set of competitive allocations, and $CU(\mathcal{P})$ for the corresponding set of utility profiles.

Existence of a competitive allocation can be derived from (much) earlier results that do not require monotonic preferences (e.g., Mas-Colell (1982, Theorem 1); see also Shafer and Sonnenschein (1975)); our main result in the next section gives instead a constructive proof.

In the normative perspective we adopt in this paper, we are only interested in efficient allocations. This is why to the usual demand property (1) we add (2) requiring demands to be *parsimonious*: each agent spends as little as possible for her competitive allocation. As already noted in Mas-Colell (1982), if we omit (2), some satiated agents in N_- may inefficiently eat some items useless to themselves but useful to others.⁵

Recall three standard normative properties of an allocation $z \in \mathcal{F}(N, A, \omega)$. It is *non-envious* if and only if (iff) $u_i(z_i) \geq u_i(z_j)$ for all i, j . It *guarantees fair share* utility iff $u_i(z_i) \geq u_i(\frac{1}{n}\omega)$ for all i . It is in the *weak core from equal split* iff for all $S \subseteq N$ and all $y \in \mathbb{R}_+^{S \times A}$ such that $y_S = \frac{|S|}{n}\omega$, there is at least one $i \in S$ such that $u_i(z_i) \geq u_i(y_i)$. When we divide goods, the competitive allocations meet these three properties, even in the much larger Arrow–Debreu preference domain. This is still true when we divide mixed items.

LEMMA 1: *A competitive allocation is efficient; it is non-envious, guarantees fair share, and is in the weak core from equal split.*

PROOF: No envy is clear. Fair share guaranteed holds because $B(p, \beta)$ contains $\frac{1}{n}\omega$. We give for completeness the standard argument for efficiency. If (z, p, β) is a competitive

⁴That is, $U \leq U'$ and $U' \in \mathcal{U}(\mathcal{P}) \implies U' = U$.

⁵For instance, $N = \{1, 2\}$, $A = \{a, b\}$, $\omega = (1, 1)$, and $u_1(z_1) = 6z_{1a} + 2z_{1b}$, $u_2(z_2) = -z_{2b}$. The inefficient allocation $z_1 = (\frac{1}{3}, 1)$ and $z_2 = (\frac{2}{3}, 0)$ meets (1) for $p = (\frac{3}{2}, \frac{1}{2})$ and $\beta = 1$. But $z'_2 = (0, 0)$ also gives zero utility to agent 2 and costs zero, so z_2 fails (2). The unique competitive division according Definition 1 is efficient: $z_1 = (1, 1)$, $z_2 = (0, 0)$, and $p = (\frac{1}{2}, \frac{1}{2})$.

division and z is Pareto-dominated by some $z' \in \mathcal{F}$, then for all $i \in N$, we must have $(p, z'_i) \geq (p, z_i)$ because otherwise i can either benefit or save money by switching to z'_i (property (2)). Since z' dominates z , some agent j strictly prefers z'_j to z_j and, therefore, z'_j is outside his budget set, that is, $(p, z'_j) > (p, z_j)$. Summing up these inequalities over all agents, we get the contradiction $(p, \omega) > (p, \omega)$. The argument for the weak core property is similar. *Q.E.D.*

REMARK 1: A competitive allocation may fail the standard core from equal split property, where coalition S blocks allocation z if it can use its endowment $\frac{|S|}{n}e^A$ to make everyone in S weakly better off and at least one agent strictly better. This is because “equal split” may give resources to agents who have no use for them.

4. MAIN RESULT

We define formally the partition of division problems alluded to in the Introduction. Given a problem \mathcal{P} , we partition N as

$$N_+ = \{i \in N \mid \exists z \in \mathcal{F} : u_i(z_i) > 0\}; \quad N_- = \{i \in N \mid \forall z \in \mathcal{F} : u_i(z_i) \leq 0\}.$$

We call agents in N_+ *attracted to the manna*, and those in N_- *repulsed by it*. All agents in N_- , and only those, are globally satiated, and for them, $z_i = 0$ is a global maximum, not necessarily unique.

The partition is determined by the relative position of the set \mathcal{U} of feasible utility profiles and the cone $\Gamma = \mathbb{R}_+^{N_+} \times \{0\}^{N_-}$, where attracted agents benefit while repulsed agents do not suffer. Write $\Gamma^* = \mathbb{R}_+^{N_+} \times \{0\}^{N_-}$ for the relative interior of Γ .

LEMMA 2: *Each problem \mathcal{P} is of (exactly) one of three types: positive if $\mathcal{U} \cap \Gamma^* \neq \emptyset$; negative if $\mathcal{U} \cap \Gamma = \emptyset$; null if $\mathcal{U} \cap \Gamma = \{0\}$.*

Note that the problem is positive if $u_i(\omega) > 0$ for at least one agent i . If we give nothing to a repulsed agent, an arbitrarily small allocation such that $u_j(z_j) > 0$ to each attracted agent j other than i , and the rest to i , the resulting utility profile is in Γ^* when z_j is small enough. The converse statement is not true: we give an example in the next section of a positive problem such that $u_i(\omega) < 0$ for all i .

Given a smooth function f and a closed convex set C , we say that $x \in C$ is a *critical point of f in C* if the upper contour of f at x has a supporting hyperplane that supports C as well:

$$\forall y \in C : \quad \partial f(x) \cdot y \leq \partial f(x) \cdot x \quad \text{and/or} \quad \forall y \in C : \quad \partial f(x) \cdot y \geq \partial f(x) \cdot x. \quad (3)$$

This holds in particular if x is a local maximum or local minimum of f in C .

In the next statement we write \mathcal{U}^{eff} for the set of efficient utility profiles, and write \mathbb{R}_-^N for the interior of \mathbb{R}_-^N .

THEOREM 1: *Competitive divisions exist in all problems \mathcal{P} . Moreover, the following statements hold:*

(i) *If \mathcal{P} is positive, the budget is $+1$; an allocation is competitive iff its utility profile maximizes the product $\prod_{N_+} U_i$ over $\mathcal{U} \cap \Gamma^*$; so $CU(\mathcal{P})$ contains a single utility profile, positive in N_+ and null in N_- .*

(ii) If \mathcal{P} is negative, the budget is -1 ; an allocation is competitive iff its utility profile is in $\mathcal{U}^{\text{eff}} \cap \mathbb{R}^N_-$ and is a critical point of the product $\prod_N |U_i|$ in \mathcal{U} ; so all utility profiles in $CU(\mathcal{P})$ are negative.

(iii) Problem \mathcal{P} has a competitive division with a zero budget iff it is null; an allocation is competitive iff its utility profile is 0.

We see that the competitive utility profiles are entirely determined by the set of feasible utility profiles: the competitive approach still has a welfarist interpretation when we divide a mixed manna.

Moreover, Theorem 1 implies that the task of dividing the manna is either good news (at least weakly) for everyone or strictly bad news for everyone.

REMARK 2: The competitive equilibrium with fixed income shares (CEFI for short) replaces in Definition 1 the common budget β by individual budgets $\theta_i\beta$, where the positive weights θ_i are independent of preferences. It is well known that in an all goods problem, this asymmetric generalization of the competitive solution obtains by maximizing the weighted product $\prod_N U_i^{\theta_i}$ of utilities, so that it preserves the uniqueness, and computational and continuity properties of the symmetric solution. The same is true of our Theorem 1, which remains valid word for word for the CEFI divisions upon raising $|U_i|$ to the power θ_i . In particular, the partition of problems in positive, negative, or null is unchanged.

5. EXAMPLES AND THE MULTIPLICITY ISSUE

We restrict attention in this section to the simple subdomain $\mathcal{L}(A)$ of $\mathcal{H}(A)$, where utilities are additive and represent linear preferences. Recall that the online platforms discussed in the Introduction work only in $\mathcal{L}(A)$, and that the companion paper provides additional normative results about competitive division in this domain.

5.1. Additive Utilities

An additive utility function is a vector $u_i \in \mathbb{R}^A$. We write $U_i = u_i \cdot z_i = \sum_A u_{ia} z_{ia}$ for the corresponding utility at allocation z_i . For agent i , item a is a good (resp. a bad) if $u_{ia} > 0$ (resp. $u_{ia} < 0$); if $u_{ia} = 0$, she is satiated with any amount of a . Given a problem \mathcal{P} , we partition A as

$$A_+ = \{a | \exists i : u_{ia} > 0\}; \quad A_- = \{a | \forall i : u_{ia} < 0\}; \quad A_0 = \{a | \max_i u_{ia} = 0\}. \quad (4)$$

We call an item in A_+ a *collective good*, an item in A_- a *collective bad*, and an item in A_0 a *neutral* item. In an efficient allocation, an item in A_+ is consumed only by agents for whom it is a good, and an item in A_0 is consumed only by agents who are indifferent to it. This partition determines the sign of competitive prices.

If (z, p, β) is a competitive division, we have

$$p_a > 0 \quad \text{if } a \in A_+; \quad p_a < 0 \quad \text{if } a \in A_-; \quad p_a = 0 \quad \text{if } a \in A_0. \quad (5)$$

The proof is simple. If the first statement fails, an agent who likes a would demand an infinite amount of it; if the second statement fails, nobody would demand b . If the third statement fails with $p_a > 0$, the only agents who demand a have $u_{ia} = 0$, which violates

parsimony (2); if it fails with $p_a < 0$, an agent such that $u_{ia} = 0$ gets an arbitrarily cheap demand by asking large amounts of a , so (2) fails again.

Suppose a collective good $a \in A_+$ is a bad for agent i , $u_{ia} < 0$, and let z be a competitive allocation. Then i consumes no a , $z_{ia} = 0$; moreover, if we only replace u_{ia} by $u'_{ia} = 0$, ceteris paribus, then z is still competitive in the new problem. This is why it is without loss of generality that in the examples below, a collective good is never a bad for anyone: $a \in A_+ \implies u_{ia} \geq 0$ for all i .

Another innocuous assumption is that the manna has one unit of each item.

5.2. Two Items

5.2.1. One Good, One Bad

The simplest mixed manna problem is easy to solve. Assume $A = \{a, b\}$ and $u_{ib} < 0 < u_{ia}$ for all i . Then label the agents so that $\frac{u_{ia}}{|u_{ib}|}$ decreases with i : agent 1 is the least averse to b relative to a . The problem is positive iff $\frac{u_{1a}}{|u_{1b}|} > 1$. In this case the competitive division gives all of b and some of a to agent 1, and the same smaller share of a to agents 2, \dots , n .

The problem is null iff $\frac{u_{1a}}{|u_{1b}|} = 1$ (agent 1 eats the entire manna) and is negative iff $\frac{u_{1a}}{|u_{1b}|} < 1$. In the latter case, the competitive division is still unique (utilitywise) and has agent 1 eating all a and some b while others eat the same smaller share of b . Here is an example with two bads and three agents:

$$\mathcal{P}: \begin{array}{cc} & a \quad b \\ u_1 & 3 \quad -4 \\ u_2 & 2 \quad -3 \\ u_3 & 1 \quad -2 \end{array} \rightarrow CE(\mathcal{P}): \begin{array}{cc} & a \quad b \\ z_1 & 1 \quad \frac{5}{6} \\ z_2 & 0 \quad \frac{1}{12} \\ z_3 & 0 \quad \frac{1}{12} \end{array}.$$

Note that in all three cases, agent 1 ends up exactly at her fair share utility level $u_i \cdot \frac{1}{n} \omega$, while the other agents get strictly more.

5.2.2. Two Bads

Computing competitive allocations is easy in this case because they are aligned in the one-dimensional set of efficient and envy-free allocations.

We assume for simplicity that all ratios $\frac{u_{ia}}{u_{ib}}$ are different, and we label the agents $\{1, \dots, n\}$ so that $\frac{u_{ia}}{u_{ib}}$ increases with i . Efficiency means that if i consumes some a and j consumes some b , then $i \leq j$; therefore, at most one agent consumes both bads. Let z^i be the allocation where the agents in $\{1, \dots, i\}$ share a (each gets $\frac{1}{i}$ unit) while those in $\{i+1, \dots, n\}$ share b . It is efficient, and all envy-free allocations sit on the line $\bigcup_{i=1}^{n-1} [z^i, z^{i+1}]$. It is easy to check that z^i is envy-free iff i and $i+1$ do not envy each other, and in that case it is a competitive allocation as well. There may be other such allocations in the open intervals $]z^i, z^{i+1}[$. The tedious but straightforward computations are detailed in Proposition 4 of the companion paper: they imply that the upper bound on the cardinality of $CU(\mathcal{P})$ is $2n - 1$.

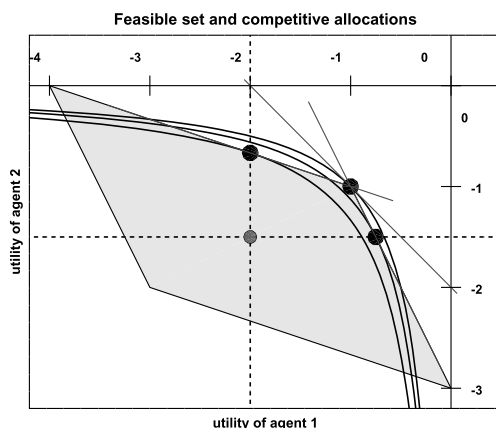


FIGURE 2.—Three big dots represent the competitive utility profiles; the smaller dot corresponds to the equal split of the manna in which both agents get their fair share utilities.

An example with $n = 2$, where $|CU(\mathcal{P})| = 3$, is

$$\begin{array}{l}
 \mathcal{P} : \begin{array}{cc} a & b \\ u_1 & -3 \quad -1, \\ u_2 & -1 \quad -2 \end{array} \\
 CE(\mathcal{P}) : \begin{array}{cc} a & b \\ z_1 & \frac{1}{3} \quad 1, \\ z_2 & \frac{2}{3} \quad 0 \end{array} \quad \begin{array}{cc} a & b \\ z'_1 & 0 \quad 1, \\ z'_2 & 1 \quad 0 \end{array} \quad \begin{array}{cc} a & b \\ z''_1 & 0 \quad \frac{3}{4}, \\ z''_2 & 1 \quad \frac{1}{4} \end{array}
 \end{array} \tag{6}$$

and the corresponding utility profiles are depicted on Figure 2. Note that agent 1 (resp. agent 2) only gets his fair share utility at z (resp. at z'').

5.3. Two Agents, Goods and Bads

Two agent problems are prominent in practice: they make for more than half of the visits on the SPLIDDIT website, and ADJUSTED WINNER is designed only for those problems. We explain after the examples why the competitive allocations are still easy to compute.

Let us add a good c to the problem \mathcal{P} from Section 5.2.2. Assuming that c is worth $\lambda > 0$ for both agents, we get the family of problems

$$\mathcal{P}(\lambda) : \begin{array}{ccc} a & c & b \\ u_1 & -3 & \lambda \quad -1. \\ u_2 & -1 & \lambda \quad -2 \end{array}$$

When λ goes from 4 to 1, the problem from positive becomes null ($\lambda = 2$) and then negative, thereby illustrating all the patterns described by Theorem 1. Figure 3 shows the competitive utility profiles for three representative values of λ : 4, 2, and 1. And Figure 2 is for $\lambda = 0$.

As noted after Lemma 2, the problem is positive if $u_i \cdot \omega > 0$ for one of $i = 1, 2$: this is true here for $\lambda > 3$. But the problem is still positive for $2 < \lambda \leq 3$, despite the fact that $u_i \cdot \omega \leq 0$ for both agents.

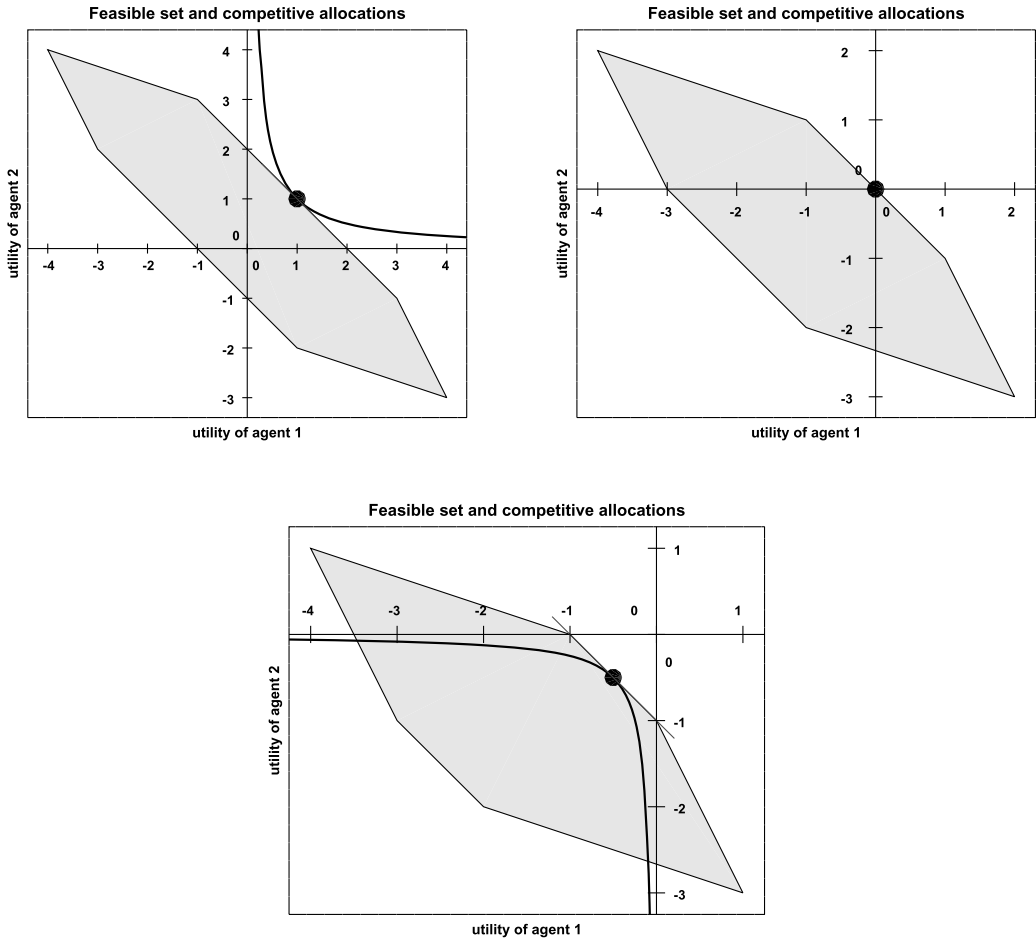


FIGURE 3.—Positive ($\lambda = 4$), null ($\lambda = 2$), and negative problems ($\lambda = 1$).

In a two agent problem, the set of efficient utility profiles is the one-dimensional frontier of a two-dimensional polytope. Recall that each item a is either a good for both agents or a bad for both, and assume for simplicity that all ratios $\frac{u_{1a}}{u_{2a}}$ are different. We label the items as $A = \{1, \dots, m\}$, ensuring that $\frac{u_{1k}}{u_{2k}}$ decreases with k . For $k \in \{0, 1, \dots, m\}$ let z^k be the allocation where agent 1 gets all goods in $\{1, \dots, k\}$ (or no good if $k = 0$) and all bads in $\{k + 1, \dots, m\}$ (or no bad if $k = m$); agent 2 eats the rest, that is, all goods in $\{k + 1, \dots, m\}$ and all bads in $\{1, \dots, k\}$. It is easy to check that the broken line $\bigcup_{k=0}^m [z^k, z^{k+1}]$ is precisely the set of efficient allocations, and it is fairly straightforward to compute all competitive allocations from the sequence of ratios $\frac{u_{1k}}{u_{2k}}$. It follows that the upper bound on $|CU(\mathcal{P})|$ is $2m - 1$ (statement (iii) in Theorem 1 of the companion paper).

5.4. Counting Competitive Allocations

In problems with three or more agents and items, it is no longer elementary to compute the set of competitive allocations. We know that the number $|CU(\mathcal{P})|$ of different

competitive utility profiles is always finite, and that its upper bound grows exponentially in n, m : $|CU(\mathcal{P})|$ can be as high as $2^{\min\{n,m\}} - 1$ if $n \neq m$, and $2^{n-1} - 1$ if $n = m$ (statement (iii) in Theorem 1 of the companion paper). A 6×5 example illustrating this fact is given in Section 8.3.

We also check in the companion paper that if $n = 2$ and/or $m = 2$, $|CU(\mathcal{P})|$ is odd in almost all problems (excluding only those where the coefficients of u satisfy certain simple equations). We conjecture that a similar statement holds for any n, m .

6. SINGLE-VALUED COMPETITIVE DIVISION RULES

Without backing up this proposal by specific normative arguments, we submit that a natural selection of $CU(\mathcal{P})$ obtains by *maximizing* the product $\prod_{i \in N} |U_i|$ of individual disutilities on the negative efficiency frontier.⁶

LEMMA 3: *If \mathcal{P} is a negative problem, the profile U^* maximizing the Nash product $\prod_{i \in N} |U_i|$ over $\mathcal{U}^{\text{eff}} \cap \mathbb{R}_-^N$ is a critical point of this product on \mathcal{U} . It is a competitive utility profile: $U^* \in CU(\mathcal{P})$.*

We prove Lemma 3 in Section 8.4 in the domain $\mathcal{H}(A)$. In the additive domain $\mathcal{L}(A)$, this selection is almost always unique.

LEMMA 4: *Fix N, A , and ω . For almost all negative problems $\mathcal{P} = (N, A, u, \omega)$ with additive utilities (with respect to (w.r.t.) the Lebesgue measure on the space $\mathbb{R}^{N \times A}$ of utility matrices), the utility profile U^* defined in Lemma 3 is unique.*

In the 6×5 example of Section 8.3, U^* is the profile treating equally the first five agents. However, in example (6) the three competitive utility profiles are $(-2, -\frac{2}{3})$, $(-1, -1)$, and $(-\frac{3}{4}, -\frac{3}{2})$, and U^* is the first one where agent 1 gets his fair share -2 . This is the typical (in fact generic) situation for two agent, two-bad problems where $|CU(\mathcal{P})| = 3$: our selection always picks one of the two extreme divisions instead of the natural compromise where both agents improve upon their fair share.

Finally, when $n = 2$ and/or $m = 2$, we mentioned in the previous subsection that $|CU(\mathcal{P})|$ is generically odd. As the corresponding profiles are arranged along a broken line, a more natural selection (again, not supported by any normative argument) picks the median profile.

Whatever the selection we end up proposing, it will exhibit the two unpalatable features to which we now turn.

7. TWO IMPOSSIBILITY RESULTS

7.1. The Continuity Issue

We show that there is no single-valued selection $\mathcal{P} \rightarrow CU(\mathcal{P})$ continuous in the utility parameters. In fact, we show a stronger discontinuity result about the much bigger correspondence of utility profiles at efficient and envy-free allocations.

Because Proposition 1 and Proposition 2 (in the next subsection) deal with axioms comparing the allocations selected at different problems, we define first division rules. For the

⁶Note that *minimizing* $\prod_{i \in N} |U_i|$ on $\mathcal{U} \cap \mathbb{R}_-^N$ picks all the boundary points where this product is null—neither a competitive allocation nor a meaningful division rule.

sake of brevity, we only give the definition in the additive domain $\mathcal{L}(A)$. Its extension to $\mathcal{H}(A)$ is straightforward.

NOTATION: When we rescale each utility u_i as $\lambda_i u_i$, the new utility matrix is written $\lambda * u$.

DEFINITION 2: A division rule f associates to every problem $\mathcal{P} = (N, A, u, \omega)$ a set of feasible allocations $f(\mathcal{P}) \subset \mathcal{F}(N, A, \omega)$ such that for any rescaling $\lambda \in \mathbb{R}_{++}^N$, we have $f(N, A, \lambda * u, \omega) = f(N, A, u, \omega)$. Moreover, f meets Pareto indifference (PI): for every \mathcal{P} and $z, z' \in \mathcal{F}(N, A, \omega)$, we have

$$\{z \in f(\mathcal{P}) \text{ and } u_i \cdot z_i = u_i \cdot z'_i \text{ for all } i\} \implies z' \in f(\mathcal{P}).$$

Property PI implies that f is entirely determined by its utility correspondence $\mathcal{P} \rightarrow F(\mathcal{P}) = \{u \cdot z | z \in f(\mathcal{P})\}$. Invariance to rescaling makes sure that division rules are ordinal constructs: they only depend on the underlying linear preferences. The competitive division rule $\mathcal{P} \rightarrow CE(\mathcal{P})$ meets Definition 2.

Abusing language, we call the division rule f *single-valued* if the corresponding mapping F is single-valued.

We call the division rule f *continuous* (CONT) if for each choice of N , A , and ω the corresponding mapping $u \rightarrow F(N, A, u, \omega)$ is upper hemicontinuous in $\mathbb{R}^{N \times A}$. If the division rule does not depend on the units of items in A ,⁷ CONT implies that $\mathcal{P} \rightarrow F(\mathcal{P})$ is also (upper hemi-) continuous in $\omega \in \mathbb{R}_+^A$.

We call the rule f *envy-free* (EVFR) if $f(\mathcal{P})$ contains at least one envy-free allocation for every problem \mathcal{P} .

PROPOSITION 1: *With two or more bads and four or more agents, no single-valued rule can be efficient, envy-free, and continuous.*

In particular, the competitive rule CE , that is a continuous and envy-free correspondence, admits no continuous single-valued selection.

Proposition 1 is Theorem 2 in the companion paper, where it follows from the fact that when the manna contains two bads, the set of envy-free and efficient allocations can have $\lfloor \frac{2n+1}{3} \rfloor$ connected components (Proposition 4). In Section 8.6, we give for completeness a four agents, two bads example illustrating the key arguments.

This incompatibility result is tight. The equal division rule, $F_i(\mathcal{P}) = \{\frac{1}{n}u_i \cdot \omega\}$ for all \mathcal{P} , meets EVFR and CONT. A single-valued selection of the competitive rule CU meets EFF and EVFR. We leave it to the reader to construct a rule meeting EFF and CONT.

7.2. Resource Monotonicity

Our last axiomatic property is often viewed as a compelling normative consequence of the common property of the resources we are dividing. Increasing in the manna the amount of a unanimous good (an item that everyone likes) or decreasing that of a unanimous bad should not hurt anyone: welfare should be co-monotonic to the quality of the common resources. Early references are Roemer (1986) and Moulin and Thomson (1988); Thomson (2010) is a recent survey.

⁷That is, for each $\lambda > 0$, the set $F(\mathcal{P})$ is unchanged if we replace ω_a by $\lambda \omega_a$ and each u_{ia} by $\frac{1}{\lambda} u_{ia}$. Clearly CU meets this property.

We say that problem \mathcal{P}' improves problem \mathcal{P} on item $a \in A$ if \mathcal{P} and \mathcal{P}' only differ in $\omega_a \neq \omega'_a$ and either $\{\omega_a < \omega'_a \text{ and } u_{ia} \geq 0 \text{ for all } i\}$ or $\{\omega_a > \omega'_a \text{ and } u_{ia} \leq 0 \text{ for all } i\}$.

RESOURCE MONOTONICITY—RM: If \mathcal{P}' improves upon \mathcal{P} on item $a \in A$, then $F(\mathcal{P}) \leq F(\mathcal{P}')$.

PROPOSITION 2:

(i) *If we divide bads between two or more agents, no single-valued rule can be efficient, resource monotonic, and guarantee fair share.*

(ii) *The competitive rule to divide goods is resource monotonic in the additive utility domain.*

The proof of statement (i) is by means of a simple two-person, two-bad example. Fix a rule F meeting EFF, RM, and GFS, and consider the problem

$$\mathcal{P}: \omega = (1, 1) \quad \text{and} \quad \begin{array}{cc} a & b \\ u_1 & -1 \quad -4. \\ u_2 & -4 \quad -1 \end{array}$$

Set $U = F(\mathcal{P})$. As $(-1, -1)$ is an efficient utility profile, one of U_1, U_2 is at least -1 , say $U_1 \geq -1$. Now let $\omega' = (\frac{1}{9}, 1)$ and pick $z' \in f(\mathcal{P}')$. By GFS and feasibility,

$$\begin{aligned} -z'_{2b} &\geq u_2 \cdot z'_2 \geq \frac{1}{2} u_2 \cdot \omega' = -\frac{13}{18}, \\ \implies z'_{1b} &\geq \frac{5}{18} \implies u_1 \cdot z'_1 = U'_1 \leq -\frac{10}{9} < U_1, \end{aligned}$$

contradicting RM. Extending this argument to the general case $n \geq 2, m \geq 2$ is straightforward.

We omit for brevity the proof of statement (ii), which is available as Proposition 2 in the companion paper and in Segal-Halevi and Sziklai (2015) for the more general cake-division model. This statement generalizes easily to positive problems when we add a unanimous good to an already positive problem.

We stress that this positive result applies only to the domain $\mathcal{L}(A)$, it does not extend to general monotone 1-homogenous utilities in $\mathcal{H}(A)$. On the latter domain, precisely the same combination of axioms as in Proposition 2 cannot be together satisfied: see Moulin and Thomson (1988) and Thomson and Kayi (2005). This makes the contrast between goods and bads problems in $\mathcal{L}(A)$ all the more stark.

8. PROOFS

8.1. Proof of Lemma 2

The three cases are clearly mutually exclusive; we check that they are exhaustive. It is enough to show that if \mathcal{U} intersects $\Gamma_{\neq 0} = \Gamma \setminus \{0\}$, then it intersects Γ^* as well. Let $z \in \mathcal{F}$ be an allocation with $u(z) \in \Gamma_{\neq 0}$ and let i_+ be an agent with $u_{i_+}(z_{i_+}) > 0$. Define a new allocation z' with $z'_{i_+} = z_{i_+} + \varepsilon \sum_{j \neq i_+} z_j$ and $z'_j = (1 - \varepsilon)z_j$ for $j \neq i_+$. By continuity, we can select a small $\varepsilon > 0$ such that $u(z') \in \Gamma_{\neq 0}$. By construction, $z'_{i_+a} > 0$ for all $a \in A$.

For any $j \in N_+ \setminus \{i_+\}$, we can find $y_j \in \mathbb{R}^A$ such that $u_j(z'_j + \delta y_j) > 0$ for small $\delta > 0$. Indeed if $u_j(z'_j)$ is positive, we can take $y_j = 0$. And if $u_j(z'_j) = 0$, assuming that y_j does

not exist implies that z'_j is a local maximum of u_j . By concavity of u_j , it is then a global maximum as well, which contradicts the definition of N_+ .

Consider an allocation z'' : $z''_{i_+} = z'_{i_+} - \delta \sum_{j \in N_+ \setminus \{i_+\}} y_j$, $z''_j = z'_j + \delta y_j$ for $j \in N_+ \setminus \{i_+\}$ and $z''_k = z'_k$ for $k \in N_-$. For small $\delta > 0$, this allocation is feasible and yields utilities in Γ^* .

8.2. Proof of Theorem 1

Throughout the proof it is convenient to consider competitive divisions (z, p, β) with arbitrary budgets $\beta \in \mathbb{R}$ (not only $\beta \in \{-1, 0, 1\}$); this clearly yields exactly the same set of competitive allocations $CE(\mathcal{P})$ and utility profiles $CU(\mathcal{P})$.

8.2.1. Positive Problems: Theorem 1, Statement (i)

Let $\mathcal{N}(V) = \prod_{i \in N_+} V_i$ be the Nash product of utilities of the attracted agents. We fix a positive problem \mathcal{P} and proceed in two steps.

STEP 1: If U maximizes $\mathcal{N}(V)$ over $V \in U \cap \Gamma^*$ and $z \in \mathcal{F}$ is such that $U = u(z)$, then z is a competitive allocation with budget $\beta > 0$.

Let \mathcal{C}_+ be the convex cone of all $y \in \mathbb{R}^{N \times A}$ with $u(y) \in \Gamma$. For any $\lambda > 0$, put

$$\mathcal{C}_\lambda = \{y \in \mathcal{C}_+ \mid \mathcal{N}(u(y)) \geq \lambda^{|N|+1}\}.$$

Since \mathcal{P} is positive, the set \mathcal{C}_λ is nonempty for any $\lambda > 0$. Continuity and concavity of utilities imply that \mathcal{C}_λ is closed and convex. Homogeneity of utilities gives $\mathcal{C}_\lambda = \lambda \mathcal{C}_1$.

Set $\lambda^* = (\mathcal{N}(U))^{\frac{1}{|N|+1}}$. The set \mathcal{C}_λ does not intersect \mathcal{F} for $\lambda > \lambda^*$, and \mathcal{C}_{λ^*} touches \mathcal{F} at z .

STEP 1.1: There exists a hyperplane H separating \mathcal{F} from \mathcal{C}_{λ^*} .

Consider a sequence λ_n converging to λ^* from above. Since \mathcal{C}_{λ_n} and \mathcal{F} are convex sets that do not intersect, they can be separated by a hyperplane H_n . The family $\{H_n\}_{n \in \mathbb{N}}$ has a limit point H . The hyperplane H separates \mathcal{F} from \mathcal{C}_{λ^*} by continuity of u . Thus there exist $q \in \mathbb{R}^{N \times A}$ and $Q \in \mathbb{R}$ such that $\sum_{i,a} q_{ia} y_{ia} \leq Q$ for $y \in \mathcal{F}$ and $\sum_{i,a} q_{ia} y_{ia} \geq Q$ on \mathcal{C}_{λ^*} . The coefficients q_{ia} will be used to define the vector of prices p .

By the construction, z maximizes $\mathcal{N}(u(y))$ over $\mathcal{B}^N(q, Q) = \{y \in \mathcal{C}_+ \mid \sum_{i,a} q_{ia} y_{ia} \leq Q\}$. Think of the latter as a budget set with agent-specific prices.

Define the vector of prices p by $p_a = \max_{i \in N} q_{ia}$ and $\mathcal{B}^*(p, Q) = \{y \in \mathcal{C}_+ \mid \sum_i p \cdot y_i \leq Q\}$. We show now that we do not need agent-specific pricing.

STEP 1.2: The allocation z maximizes $\mathcal{N}(u(y))$ over $y \in \mathcal{B}^*(p, Q)$.

It is enough to show the double inclusion $z \in \mathcal{B}^*(p, Q) \subset \mathcal{B}^N(q, Q)$. The second one is obvious since $\sum_{i,a} y_{ia} p_a \leq Q$ implies $\sum_{i,a} y_{ia} q_{ia} \leq Q$. Let us check the first inclusion. Taking into account that $z \in \mathcal{F}$ and $\sum_{i,a} q_{ia} z_{ia} \leq Q$ for $y \in \mathcal{F}$, we get

$$\sum_i p \cdot z_i = \sum_a p_a \sum_i z_{ia} = \sum_a p_a = \sum_a \max_i q_{ia} = \max_{y \in \mathcal{F}} \sum_{i,a} q_{ia} y_{ia} \leq Q.$$

STEP 1.3: The triple (z, p, β) is a competitive division for some $\beta > 0$.

Consider an agent i from N_+ . Check that the bundle z_i belongs to his competitive demand $d_i(p, \beta_i)$, where $\beta_i = p \cdot z_i$. Indeed if there exists $z'_i \in \mathbb{R}_+^A$ such that $p \cdot z'_i \leq \beta_i$ and $u_i(z'_i) > u_i(z_i)$, then switching the consumption of agent i from z_i to z'_i gives an allocation in $\mathcal{B}^*(p, Q)$ and increases the Nash product, contradicting Step 1.2. Note that $\beta_i > 0$ for $i \in N_+$ because otherwise we can take $z'_i = 2z_i$. Check now that z_i is parsimonious: it minimizes $p \cdot y_i$ over $d_i(p, \beta_i)$. If not, pick $y_i \in d_i(p, \beta_i)$ with $p \cdot y_i < p \cdot z_i$. Then for δ small enough and positive, the bundle $z'_i = (1 + \delta)y_i$ meets $p \cdot z'_i \leq \beta_i$ and $u_i(z'_i) > u_i(z_i)$.

We use now the classic equalization argument (Eisenberg (1961)) to check that β_i does not depend on $i \in N_+$. We refer to the fact that the geometric mean is below the arithmetic mean as *the inequality of means*.

Assume $\beta_i \neq \beta_j$ and consider a new allocation z' , where the budgets of i and j are equalized: $z'_i = \frac{\beta_i + \beta_j}{2\beta_i} z_i$ and $z'_j = \frac{\beta_i + \beta_j}{2\beta_j} z_j$. This allocation belongs to $\mathcal{B}^*(p, Q)$ and homogeneity of utilities implies

$$\mathcal{N}(u(z')) = \mathcal{N}(U) \left(\frac{\beta_i + \beta_j}{2\beta_i} \right) \left(\frac{\beta_i + \beta_j}{2\beta_j} \right).$$

Now the (strict) inequality of means gives $\frac{\beta_i + \beta_j}{2} > \sqrt{\beta_i \beta_j}$; therefore, $\mathcal{N}(u(z')) > \mathcal{N}(U)$, contradicting the optimality of z . Denote the common value of β_i by β .

Turning finally to the repulsed agents, we check that for any $i \in N_-$ there is no z'_i such that $u_i(z'_i) = 0$ and $\beta'_i = p \cdot z'_i < p \cdot z_i = \beta_i$, that is, i cannot decrease his spending. Assuming that z'_i exists, we can construct an allocation $z' \in \mathcal{B}^*(p, Q)$, where agent i switches to z'_i , consumption of other agents from N_- remains the same, and $z'_j = z_j \frac{N_+ \beta + \beta_i - \beta'_i}{N_+ \beta}$ for $j \in N_+$. In other words, money saved by i is redistributed among positive agents. By homogeneity, $\mathcal{N}(u(z')) > \mathcal{N}(U)$ —a contradiction. A corollary is that β_i must be zero: take $z'_i = 0$ if $\beta_i > 0$ and $z'_i = 2z_i$ if β_i is negative. At z_i agent i reaches his maximal welfare of zero. Therefore, if i can afford z_i , then z_i is in the demand set. Since the price β_i of z_i is zero, we conclude that $z_i \in d_i(p, \beta)$. The proof of Step 1.3 and of Step 1 is complete.

STEP 2: If (z, p, β) is a competitive division, then $\beta > 0$, and $U = u(z)$ belongs to $\mathcal{U} \cap \Gamma^*$ and maximizes \mathcal{N} over this set.

Check first that $\beta > 0$. If $\beta \leq 0$, the budget set $B(p, \beta)$ contains z_i and $2z_i$ for all i ; therefore $u_i(2z_i) \leq u_i(z_i)$ implies $U_i \leq 0$. Then U is Pareto-dominated by any $U' \in \mathcal{U} \cap \Gamma^*$, contradicting the efficiency of z (Lemma 1).

Now $\beta > 0$ implies U belongs to Γ^* : every $i \in N_+$ has a y_i with $u_i(y_i) > 0$ and can afford δy_i for small enough $\delta > 0$; every $i \in N_-$ can afford $y_i = 0$, hence $u_i(z_i) = 0$ and $p \cdot z_i \leq 0$ (by (2)).

Consider $U' = u(z')$ that maximizes \mathcal{N} over $\mathcal{U} \cap \Gamma^*$. For any $i \in N_+$ his spending $\beta'_i = p \cdot z'_i$ must be positive. Otherwise $\delta z'_i \in B(p, \beta)$ for any $\delta > 0$ and agent i can reach unlimited welfare. Similarly $\beta'_i < 0$ for $i \in N_-$ implies $\delta z'_i \in d_i(p, \beta)$ for any $\delta > 0$, so the spending in $d_i(p, \beta)$ is arbitrarily low, in contradiction to parsimony (2).

For attracted agents, $\frac{\beta}{\beta'_i} z'_i \in B(p, \beta)$ gives $\frac{\beta}{\beta'_i} U'_i = u_i(\frac{\beta}{\beta'_i} z'_i) \leq U_i$. Therefore if U is not a maximizer of \mathcal{N} , we have

$$\mathcal{N}(U) < \mathcal{N}(U') \leq \mathcal{N}(U) \prod_{i \in N_+} \frac{\beta'_i}{\beta} \implies 1 < \left(\prod_{i \in N_+} \frac{\beta'_i}{\beta} \right)^{\frac{1}{|N_+|}} \leq \frac{\sum_{i \in N_+} \beta'_i}{|N_+| \beta},$$

where we use again the inequality of means. Now we get a contradiction from

$$\sum_{i \in N_+} \beta'_i \leq \sum_{i \in N} \beta'_i = p \cdot \omega = \sum_{i \in N} p \cdot z_i \leq \sum_{i \in N_+} \beta + \sum_{i \in N_-} 0 = |N_+| \beta.$$

8.2.2. Negative Problems: Theorem 1, Statement (ii)

The proof is simpler because we do not need to distinguish agents from N_+ and N_- . We define the Nash product for negative problems by $\mathcal{N}(V) = \prod_{i \in N} |V_i|$ and focus now on its critical points in \mathcal{U}^{eff} . We start by the variational characterization of such points. If $V \in \mathbb{R}_{=}^N$, we have $\frac{\partial}{\partial V_i} \mathcal{N}(V) = \frac{1}{V_i} \mathcal{N}(V)$. Therefore $U \in \mathcal{U} \cap \mathbb{R}_{=}^N$ is a critical point of \mathcal{N} on \mathcal{U} that lies on \mathcal{U}^{eff} iff

$$\sum_{i \in N} \frac{U'_i}{|U_i|} \leq -|N| \quad \text{for all } U' \in \mathcal{U}. \quad (7)$$

The choice of the sign in this inequality is determined by efficiency. Set $\varphi_U(U') = \sum_{i \in N} \frac{U'_i}{|U_i|}$: inequality (7) says that $U' = U$ maximizes $\varphi_U(U')$ on \mathcal{U} .

We fix a negative problem \mathcal{P} and proceed in two steps.

STEP 1: If a utility profile $U \in \mathcal{U}^{\text{eff}} \cap \mathbb{R}_{=}^N$ is a critical point of \mathcal{N} on \mathcal{U} , then any $z \in \mathcal{F}$ implementing U is a competitive allocation with budget $\beta < 0$.

By (7) for any $y \in \mathcal{F}$, we have $\varphi_U(u(y)) \leq -|N|$. Define

$$\mathcal{C}_\lambda = \{y \in \mathbb{R}_+^{N \times A} \mid \varphi_U(u(y)) \geq \lambda\}.$$

For $\lambda \leq 0$, it is nonempty (it contains 0), closed, and convex. For $\lambda > -|N|$, the set \mathcal{C}_λ does not intersect \mathcal{F} and for $\lambda = -|N|$, it touches \mathcal{F} at z . Consider a hyperplane $\sum_{i,a} q_{ia} y_{ia} = Q$ separating \mathcal{F} from $\mathcal{C}_{-|N|}$ and fix the sign by assuming $\sum_{i,a} q_{ia} y_{ia} \leq Q$ on \mathcal{F} (existence follows as in Step 1.1 for positive problems). By the construction, z maximizes $\varphi_U(u(y))$ on $\mathcal{B}^N(q, Q) = \{y \in \mathbb{R}_+^{N \times A} \mid \sum_{i,a} q_{ia} y_{ia} \leq Q\}$. Defining prices by $p_a = \max_{i \in N} q_{ia}$ and mimicking the proof of Step 1.2 for positive problems we obtain that z belongs to $\mathcal{B}^*(p, Q) = \{y \in \mathbb{R}_+^{N \times A} \mid \sum_{i,a} p_a y_{ia} \leq Q\}$ and maximizes $\varphi_U(u(y))$ there.

We check now that z is a competitive allocation with negative budget. For any agent $i \in N$, the bundle z_i belongs to his demand $d_i(p, \beta_i)$ (as before $\beta_i = p \cdot z_i$). If not, i can switch to any $z'_i \in B(p, \beta_i)$ with $u_i(z'_i) > U_i$, thus improving the value of φ_U and contradicting the optimality of z . The maximal spending β_i must be negative; otherwise i can afford $y_i = 0$ and $u_i(z_i) < u_i(y_i)$. If there is some $z'_i \in d_i(p, \beta_i)$ such that $p \cdot z'_i < \beta_i$, the bundle $z''_i = \frac{\beta_i}{p \cdot z'_i} z'_i$ is still in $B(p, \beta_i)$ and $u_i(z''_i) > U_i$: therefore $p \cdot z_i = \beta_i$ and z_i is parsimonious ((2)).

Finally, $\beta_i = \beta_j$ for all $i, j \in N$. If $\beta_i \neq \beta_j$, we use an *unequalization* argument dual to that in Step 1.3 for positive problems. Assume, for instance, $\beta_i > \beta_j \Leftrightarrow |\beta_i| < |\beta_j|$ and

define z' from z by changing only z'_i to $\frac{1}{2}z_i$ and z'_j to $\frac{2\beta_j + \beta_i}{2\beta_j}z_j$. Clearly $z' \in \mathcal{B}^*(p, Q)$ and we compute

$$\varphi_U(u(z')) - \varphi_U(U) = -\frac{1}{2} - \frac{2\beta_j + \beta_i}{2\beta_j} + 2 = \frac{1}{2} - \frac{\beta_i}{2\beta_j} > 0.$$

But we showed that z maximizes $\varphi_U(u(y))$ in $\mathcal{B}^*(p, Q)$ —a contradiction.

STEP 2: If (z, p, β) is a competitive division, then $\beta < 0$ and the utility profile $U = u(z)$ is a critical point of the Nash product on U that belongs to $\mathcal{U}^{\text{eff}} \cap \mathbb{R}^N_-$.

Check first that $\beta < 0$. If not, each agent can afford $y_i = 0$, so $U_i \geq 0$ for all i , which is impossible in a negative problem. Assume next $U_i \geq 0$ for some i : we have $2z_i \in B(p, \beta)$, $u_i(2z_i) \geq u_i(z_i)$, and $p \cdot (2z_i) < p \cdot z_i$, which contradicts (1) and/or (2) in Definition 1. Therefore U belongs to \mathbb{R}^N_- . Finally $p \cdot z_i < \beta$ would imply $u_i(z_i) < u_i(\lambda z_i)$ for $\lambda \in [0, 1[$ and $\lambda z_i \in B(p, \beta)$ for λ close enough to 1—a contradiction. Summarizing, we have shown $U \in \mathcal{U}^{\text{eff}} \cap \mathbb{R}^N_-$ and $p \cdot z_i = \beta < 0$ for all i .

To prove that U is a critical point it is enough to check that it maximizes $\varphi_U(u(y))$ on \mathcal{F} . Fix $z' \in \mathcal{F}$, set $U' = u(z')$, and set $p \cdot z'_i = \beta'_i$. To show $\varphi_U(U') \leq \varphi_U(U)$, we will prove

$$U'_i \leq \frac{\beta'_i}{\beta} U_i \quad \text{for all } i. \quad (8)$$

This holds if $\beta'_i < 0$ because $\frac{\beta}{\beta'_i} z'_i \in B(p, \beta)$ so $\frac{\beta}{\beta'_i} U'_i = u_i(\frac{\beta}{\beta'_i} z'_i) \leq U_i$. If $\beta'_i \geq 0$, we set $z''_i = \alpha z'_i + (1 - \alpha)z_i$, where $\alpha > 0$ is small enough that $p \cdot z''_i < 0$. We just showed (8) holds for $u_i(z''_i)$; therefore,

$$u_i(z''_i) \leq \frac{p \cdot z''_i}{\beta} U_i = \alpha \frac{\beta'_i}{\beta} U_i + (1 - \alpha) U_i.$$

Concavity of u_i gives $\alpha U'_i + (1 - \alpha) U_i \leq u_i(z''_i)$ and the proof of (8) is complete. Now we sum up these inequalities and reach the desired conclusion:

$$\varphi_U(U') = \sum_{i \in N} \frac{U'_i}{|U_i|} \leq -\frac{\sum_{i \in N} \beta'_i}{\beta} = -\frac{\sum_{i \in N} p \cdot z'_i}{\beta} = -\frac{p \cdot \omega}{\beta} = -|N| = \varphi_U(U).$$

8.2.3. Null Problems: Theorem 1, Statement (iii)

The proof resembles that for positive problems, as we must distinguish N_+ from N_- , but the Nash product no longer plays a role. Fix a null problem \mathcal{P} .

STEP 1: Any $z \in \mathcal{F}$ such that $u(z) = 0$ is competitive with $\beta = 0$.

Suppose first that all agents are repulsed, $N = N_-$. Then $u_i(y_i) \leq 0$ for all $i \in N$ and $y_i \in \mathbb{R}^A_+$, and $(z, 0, 0)$ is a competitive division: everybody has zero money, all bundles are free, and all agents achieve the best possible welfare with the smallest possible spending. We assume from now on $N_+ \neq \emptyset$.

Define $\psi(y) = \min_{i \in N_+} u_i(y_i)$ for $y \in \mathbb{R}^{N \times A}_+$ and the sets $\mathcal{C}_\lambda = \{y \in \mathcal{C}_+ \mid \psi(y) \geq \lambda\}$, where $\mathcal{C}_+ = \{y \in \mathbb{R}^{N \times A}_+ \mid u(y) \in \Gamma\}$ (as in the positive proof). For $\lambda \geq 0$ the set \mathcal{C}_λ is nonempty,

closed, and convex. If $\lambda > 0$, the sets \mathcal{C}_λ and \mathcal{F} do not intersect. As in Step 1.1 of the positive proof, we construct a hyperplane separating \mathcal{F} and \mathcal{C}_0 , and define the set $\mathcal{B}^N(q, Q)$, the vector of prices p , and the set $\mathcal{B}^*(p, Q)$. Similarly we check that the allocation z maximizes $\psi(y)$ over $y \in \mathcal{B}^*(p, Q)$, and that $\psi(z) = 0$.

We set $\beta_i = p \cdot z_i$ and show that $(z, p, 0)$ is a competitive division in three substeps.

STEP 1.1: For all $i \in N$ and $x_i \in \mathbb{R}_+^4$, $p \cdot x_i < \beta_i \implies u_i(x_i) < 0$.

Suppose $p \cdot x_i < \beta_i$ and $u_i(x_i) \geq 0$ for some $i \in N_+$. For each j in N_+ , pick a bundle y_j^+ such that $u_j(y_j^+) > 0$ and construct the allocation z' as $z'_i = x_i + \delta y_i^+$; $z'_j = z_j + \delta y_j^+$ for any other $j \in N_+$; $z'_j = z_j$ for $j \in N_-$. If $\delta > 0$ is small enough, $z' \in \mathcal{B}^*(p, Q)$ and for any $j \in N_+$, we have $u_j(z'_j) > 0$ by concavity and homogeneity of u_j . For instance,

$$\frac{1}{2}u_i(z'_i) = u_i\left(\frac{1}{2}x_i + \frac{1}{2}\delta y_i^+\right) \geq \frac{1}{2}u_i(x_i) + \frac{1}{2}\delta u_i(y_i^+) > 0.$$

Therefore $\psi(z') > 0$, contradicting the optimality of z .

The proof when $p \cdot x_i < \beta_i$ and $u_i(x_i) \geq 0$ for some $i \in N_-$ is similar and is left to the reader.

STEP 1.2: We have $\beta_i = 0$ for all $i \in N$.

If $\beta_i > 0$, then $x_i = 0$ is such that $p \cdot x_i < \beta_i$ and $u_i(x_i) = 0$, which we just ruled out. If $\beta_i < 0$, then $p \cdot (2z_i) < \beta_i$ yet $u_i(2z_i) = 0$, contradicting Step 1.1.

From Steps 1.1 and 1.2 we see that for all i , if $y_i \in d_i(p, 0)$, then $p \cdot y_i = 0$: so if we show $z_i \in d_i(p, 0)$, the parsimony property (2) is automatically satisfied. Therefore our next substep completes the proof of Step 1.

STEP 1.3: We have $z_i \in d_i(p, 0)$ for all $i \in N$.

For $i \in N_-$ this is obvious since such an agent reaches his maximal welfare $u_i = 0$. Pick now $i \in N_+$ and assume $z_i \notin d_i(p, 0)$. Then $d_i(p, 0)$ contains some y_i with $u_i(y_i) > 0$. Let w be a bundle with negative price. Such a bundle exists since $p \cdot \omega = \sum_{i \in N} \beta_i = 0$ and $p \neq 0$. Hence the bundle $x_i = y_i + \delta w$ with small enough $\delta > 0$ has negative price $p \cdot x_i < 0$ and $u_i(x_i) > 0$ —a contradiction.

STEP 2: If (z, p, β) is a competitive division, then $u(z) = 0$ and (z, p, β') with $\beta' = 0$ is also competitive.

If $\beta < 0$, we have $u_i(z_i) < 0$ for all $i \in N$. Otherwise $u_i(z_i) \geq 0$ and $p \cdot z_i < 0$ implies as before that $z'_i = 2z_i$ improves U_i (at least weakly) while remaining in $B(p, \beta)$ and lowering i 's spending. But $U \in \mathbb{R}_+^N$ is not efficient in a null problem.

Thus $\beta \geq 0$, hence $u_i(z_i) \geq 0$ for all $i \in N$ because the bundle 0 is in the budget set. The problem is null and, therefore, $u(z) = 0$, implying $0 \in d_i(p, \beta)$ and by parsimony (2) $p \cdot z_i \leq 0$, for all i . Hence $z_i \in d_i(p, 0)$; therefore, $(z, p, 0)$ is clearly a competitive division.

8.3. *An Exponential Number of Competitive Divisions*

We give an example where $n = 6, m = 5$, where $|CU(\mathcal{P})| = 2^5 - 1 = 31$. Set $N = \{1, 2, 3, 4, 5, 6\}$ and $A = \{a, b, c, d, e\}$, and consider the problem

	a	b	c	d	e
u_1	-1	-3	-3	-3	-3
u_2	-3	-1	-3	-3	-3
u_3	-3	-3	-1	-3	-3
u_4	-3	-3	-3	-1	-3
u_5	-3	-3	-3	-3	-1
u_6	-1	-1	-1	-1	-1

There is one symmetric competitive division with uniform price $\frac{6}{5}$ for each bad: it gives $\frac{5}{6}$ of her preferred bad to each agent $1, \dots, 5$, while agent 6 eats $\frac{1}{6}$ of every bad, precisely his fair share. Now for each strict subset of the first five agents, for instance, $\{3, 4, 5\}$, there is a competitive allocation where each such agent eats “her” bad in full, while agent 1 shares the other bads with the other agents:

	a	b	c	d	e
z_1	2/3	0	0	0	0
z_2	0	2/3	0	0	0
z_3	0	0	1	0	0
z_4	0	0	0	1	0
z_5	0	0	0	0	1
z_6	1/3	1/3	0	0	0

Here prices are $p = -(\frac{3}{2}, \frac{3}{2}, 1, 1, 1)$.

This construction can be adjusted for each nontrivial subset of the first five agents. Then agent 6's utility goes from -1 (his fair share) to $-\frac{1}{2}$, when he shares a single bad with a single other agent; utilities of other agents vary also between -1 and $-\frac{1}{2}$.

8.4. *Proof of Lemma 3*

We have $u_i(\omega) < 0$ for every $i \in N$, else the allocation z with $z_i = \omega$ and $z_j = 0$ for $j \neq i$ yields utilities in Γ .

Consider the set of utility profiles dominated by $\mathcal{U} \cap \mathbb{R}_-^N$: $\mathcal{U}_\leq = \{U \in \mathbb{R}_-^N \mid \exists U' \in \mathcal{U} \cap \mathbb{R}_-^N : U \leq U'\}$. This set is closed and convex, and contains all points in \mathbb{R}_-^N that are sufficiently far from the origin. Indeed, any $U \in \mathbb{R}_-^N$ such that $U_N < \min_i u_i(\omega)$, where $U_N = \sum_i U_i$, is dominated by the utility profile $z : z_i = \frac{U_i}{U_N} \omega, i \in N$.

Fix $\lambda \geq 0$ and consider the upper contour of the Nash product at λ : $C_\lambda = \{U \in \mathbb{R}_-^N \mid \prod_N |U_i| \geq \lambda\}$. For sufficiently large λ , the closed convex set C_λ is contained in \mathcal{U}_\leq . Let λ^* be the minimal λ with this property. Negativity of \mathcal{P} implies that \mathcal{U}_\leq is bounded away from 0 so that λ^* is strictly positive. By definition of λ^* , the set C_{λ^*} touches the boundary of \mathcal{U}_\leq at some U^* with strictly negative coordinates. Let H be a hyperplane supporting \mathcal{U}_\leq at U^* . By the construction, this hyperplane also supports C_{λ^*} ; therefore, U^* is a critical point of the Nash product on \mathcal{U}_\leq , that is, U^* maximizes $\sum_{i \in N} \frac{U_i}{|U_i|}$ over all $U \in \mathcal{U}_\leq$. So U^* belongs to the Pareto frontier of \mathcal{U}_\leq , which is contained in the Pareto frontier of \mathcal{U} . Thus U^* is a critical point of the Nash product on \mathcal{U} and belongs to $\mathcal{U}^{\text{eff}} \cap \mathbb{R}_-^N$. By the construction, any

U in the interior of C_{λ^*} is dominated by some $U' \in \mathcal{U} \cap \mathbb{R}^N_-$, so U^* maximizes the Nash product on $\mathcal{U}^{\text{eff}} \cap \mathbb{R}^N_-$.

8.5. Proof of Lemma 4

In the previous proof, note that the supporting hyperplane H to \mathcal{U} at U^* is unique because it is also a supporting hyperplane to C_{λ^*} that is unique. Hence, if \mathcal{U} is a polytope, U^* belongs to a face of maximal dimension.

When utilities are additive, both sets \mathcal{U} and \mathcal{U}_{\leq} are polytopes. Let $D \subset \mathbb{R}^{N \times A}$ be the set of all u such that the problem (N, A, u) is negative and U^* is not unique. By the above remark if $u \in D$, then for some $\lambda > 0$, the set \mathcal{U}_{\leq} has at least two faces F and F' of maximal dimension that are tangent to the surface S_{λ} : $\prod_{i \in N} |U_i| = \lambda$, $U \in \mathbb{R}^N_-$. The condition that F is tangent to S_{λ} fixes λ . The set of all hyperplanes tangent to a fixed surface S_{λ} has dimension $|N| - 1$ (for every point on S there is one tangent hyperplane) though the set of all hyperplanes in \mathbb{R}^N is $|N|$ -dimensional. Hence tangency of F' and S_{λ} cuts one dimension. So D is contained in a finite union of algebraic surfaces and, therefore, has Lebesgue measure zero.

8.6. Proof of Proposition 1

Recall the complete proof is in Proposition 4 and Theorem 2 of the companion paper. We fix $N = \{1, 2, 3, 4\}$ and $A = \{a, b\}$, and construct two problems. The set $\mathcal{A}(0)$ of efficient and envy-free allocations in the first problem $\mathcal{P}(0)$ has three connected components \mathcal{B}_i , $i = 1, 2, 3$. By a symmetry argument, it is without loss of generality to assume that an efficient and envy-free rule picks an allocation in \mathcal{B}_2 or \mathcal{B}_3 . Then as we move linearly from $\mathcal{P}(0)$ to $\mathcal{P}(1)$, the components \mathcal{B}_2 and \mathcal{B}_3 of \mathcal{A} disappear, without ever intersecting \mathcal{B}_1 , until eventually $\mathcal{A}(1) = \mathcal{B}_1$. This forces a discontinuity in the rule. Define

	a	b
u_1	1	5
u_2	1	4
u_3	4	1
u_4	5	1

$\mathcal{P}(0) :$

	a	b
u_1	1	5
u_2	1	4
u_3	1	2
u_4	2	1

$\mathcal{P}(1) :$

If an allocation z is efficient, then at most one agent i eats both bads. Suppose it is agent 1: then if z is also envy-free, it takes the form

	a	b
z_1	1	$1 - 3x$
z_2	0	x
z_3	0	x
z_4	0	x

$z =$

$\frac{3}{10} \leq x \leq \frac{5}{16}.$

This interval in \mathcal{F} is \mathcal{B}_1 , the first component of $\mathcal{A}(0)$, in which $x = \frac{3}{10}$ yields the competitive allocation z^1 . Exchanging the roles of a and b and of agents i and $5 - i$, we find symmetrically the component \mathcal{B}_3 ,

	a	b
z_1	x	0
z_2	x	0
z_3	x	0
z_4	$1 - 3x$	1

$z =$

$\frac{3}{10} \leq x \leq \frac{5}{16},$

with another competitive allocation z^2 at $x = \frac{3}{10}$. The third component \mathcal{B}_2 is itself symmetric around the symmetric competitive allocation z^3 , where agents 1 and 2 split a equally, while 3 and 4 split b . Unlike the two other components, \mathcal{B}_2 is of dimension 2: half of \mathcal{B}_2 is the set of allocations

$$\begin{array}{ccc} & a & b \\ z_1 & x & 0 \\ z = z_2 & 1 - x & 1 - 2y, \quad \text{where } x + 5y \leq 3, 4x + 3y \leq 5, 2x + 8y \geq 5; \\ z_3 & 0 & y \\ z_4 & 0 & y \end{array}$$

the other half obtains by exchanging the roles of a and b and of agents i and $5 - i$. So $\mathcal{A}(0) = \bigcup_{i=1}^3 \mathcal{B}_i$.

Turning to $\mathcal{P}(1)$, we see that the above allocation z^1 is still competitive, and the corresponding connected component of $\mathcal{A}(1)$ is still the interval \mathcal{B}_1 . However, there is no other competitive allocation: $\mathcal{A}(1) = \mathcal{B}_1$. Finally we consider the interval of problems $\mathcal{P}(\lambda)$ connecting our two problems

$$\begin{array}{ccc} & a & b \\ \mathcal{P}(\lambda) : u_1 & 1 & 5 \\ u_2 & 1 & 4 \\ u_3 & 4 - 3\lambda & 1 + \lambda \\ u_4 & 5 - 3\lambda & 1 \end{array}, \quad 0 \leq \lambda \leq 1,$$

and check that \mathcal{B}_1 remains a component for all λ while \mathcal{B}_2 and \mathcal{B}_3 shrink and eventually disappear after $\lambda = \frac{3}{4}$.

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